

## Error Bounds for Spline-on-Spline Interpolation

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Bounds for the uniform norm of the errors in the second and third derivatives of cubic interpolating splines-on-splines are derived. These bounds encompass the case when approximate rather than exact derivatives are used at the endpoints. Furthermore, it is shown that, for a uniform mesh, spline-on-spline techniques lead to an accuracy of one additional order at the knots. Results of some computational experiments are given.

### 1. INTRODUCTION

A popular method of calculating derivatives of a function from its values on a given set of knots uses splines. In this paper we discuss an alternative strategy—the “spline-on-spline” technique. There is computational evidence that such a procedure yields better results than the traditional process using a single spline. For example, Ahlberg *et al.* observed [4] that computationally the spline-on-spline method gives excellent results for the second derivative of  $\sin x$ .

The spline-on-spline technique can be described as follows: suppose we have an interval  $[a, b]$  partitioned by a mesh  $\pi$ . In order to map a given function  $x(t)$  into an approximant of its first derivative, we interpolate the function by a cubic spline  $s(t)$  over the mesh, requiring  $s(t)$  to take on slopes  $\sigma_1, \sigma_2$  at the respective endpoints of  $[a, b]$ . Then  $x(t)$  is replaced by the spline's derivative:  $x(t) \rightarrow (d/dt) s(t)$ . A repetition of the procedure gives us an approximant of  $x''(t)$ , and so on through higher orders of the derivative if desired.

The aim of the paper is to furnish error bounds for the spline-on-spline

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method. These are given by inequalities (29), (30), (31) below. They attain a simple form provided the mesh  $\pi$  is uniform, i.e.,

$$|x^{(p)}(t) - s_p(t)| \leq \frac{1}{24} \sum_{j=1}^p \left(\frac{9}{2}\right)^{p-j} \|x^{(3+j)}\| h^{3+j-p} + \sum_{j=1}^p \left(\frac{9}{2}\right)^{p-j} |d_j| h^{j-p},$$

where  $s_p$  stands for the approximant to  $x^{(p)}$ ,  $h$  is the mesh gauge and  $d_j$  is the two-dimensional vector of errors in the prescription of endpoint slopes for interpolation purposes; thus,

$$|d_j| = \max\{|x^{(j)}(a) - \sigma_1^{(j)}|, |x^{(j)}(b) - \sigma_2^{(j)}|\}.$$

In the second section of the paper we give the following appraisal of "discretized error":

$$|x^{(p)}(t_i) - s_p(t_i)| \leq \frac{1}{60} \sum_{j=1}^p 3^{p-j} \|x^{(4+j)}\| h^{4+j-p} + \sum_{j=1}^p 3^{p-j} |d_j| h^{j-p}.$$

Here  $t_i$  is any one of the knots (the mesh being assumed uniform).

In the last section results of some numerical experiments are presented.

## 2. UNIFORM NORM ESTIMATES

Let  $R^m$  be the real,  $m$ -dimensional Euclidean space with norm  $|\xi| = \max_{1 \leq i \leq m} |\xi_i|$ .

For a fixed mesh  $\pi: a = t_0 < t_1 < \dots < t_n = b$ , let  $h_i = t_i - t_{i-1}$ ,  $i = 1, 2, \dots, n$ ,  $h = \max_{1 \leq i \leq n} h_i$ ,  $k = \min_{1 \leq i \leq n} h_i$ ,  $l = \min\{h_1, h_n\}$ , and  $\beta = h/k$ .

Also, for  $i = 1, 2, \dots, n - 1$ , denote

$$\lambda_i = h_{i+1}(h_i + h_{i+1})^{-1}, \quad \mu_i = 1 - \lambda_i.$$

Let  $C$  be the space  $C[a, b]$ . For an integer  $p \geq 0$ , let  $C^p = \{x: x^{(p)} \in C\}$ ; also, let  $C_\pi^p$  be the space of all  $x \in C$  such that  $x^{(p-1)} \in C$  has a bounded derivative  $x^{(p)}$  on  $[a, b] - \pi'$ , with  $\pi' = \{t_1, t_2, \dots, t_{n-1}\}$ . Moreover, if  $x(t)$  is defined and bounded on  $[a, b] - \pi'$ , we let  $\|x\| = \sup\{|x(t)|: t \in [a, b] - \pi'\}$ .

Define an operator  $\mathcal{S}: C \times R^2 \rightarrow C^2$  as follows: for  $x \in C$  and  $\sigma = (\sigma_1, \sigma_2) \in R^2$ , let  $\mathcal{S}(x, \sigma)$  be the unique cubic spline associated with  $x$  and the mesh  $\pi$  such that  $[\mathcal{S}(x, \sigma)]'(t_0) = \sigma_1$  and  $[\mathcal{S}(x, \sigma)]'(t_n) = \sigma_2$ .

Furthermore, for  $x \in C$  we put  $\langle x \rangle = (x(t_0), x(t_n)) \in R^2$ , and, for  $x \in C^1$ ,  $Sx = \mathcal{S}(x, \langle x' \rangle)$ —the complete cubic spline.

To simplify the notation in subsequent considerations, we put  $s = \mathcal{S}(x, \sigma)$ ,  $s_i^j = s^{(j)}(t_i)$ ,  $i = 0, 1, \dots, n$ ,  $j = 0, 1, 2$ , and  $s^j = (s_0^j, s_1^j, \dots, s_n^j) \in R^{n+1}$ ,  $j = 0, 1, 2$ . Thus,  $s_i^0 = x(t_i)$  for  $i = 0, 1, \dots, n$  and  $s_0^1 = \sigma_1$ ,  $s_n^1 = \sigma_2$ .

LEMMA 1. (a) For  $x \in C^1_\pi$ ,

$$|s^1| \leq 3 \|x'\| + |\sigma|. \tag{1}$$

(b) For  $x \in C$  and a uniform mesh  $\pi$ ,

$$|s^1| \leq 3h^{-1} |s^0| + |\sigma|. \tag{2}$$

*Proof.* (a) For  $i = 1, 2, \dots, n - 1$  we have [4, p. 13]

$$\lambda_i s^1_{i-1} + 2s^1_i + \mu_i s^1_{i+1} = 3\lambda_i h_i^{-1} (s^0_i - s^0_{i-1}) + 3\mu_i h_{i+1}^{-1} (s^0_{i+1} - s^0_i) \tag{3}$$

with  $s^0_0 = \sigma_1, s^0_n = \sigma_2$ . For  $i = 2, 3, \dots, n - 2$ , (3) yields

$$\begin{aligned} 2 |s^1_i| &\leq \lambda_i |s^1_{i-1}| + \mu_i |s^1_{i+1}| + 3\lambda_i |h_i^{-1} (s^0_i - s^0_{i-1})| + 3\mu_i |h_{i+1}^{-1} (s^0_{i+1} - s^0_i)| \\ &\leq |s^1| + 3 \max_{i=2, \dots, n-1} |h_i^{-1} (s^0_i - s^0_{i-1})|. \end{aligned}$$

Similarly, for  $i = 1, n - 1$  we get from (3),

$$2 |s^1_1| \leq |s^1| + 3 \max_{i=1,2} |h_i^{-1} (s^0_i - s^0_{i-1})| + |\sigma_1|,$$

$$2 |s^1_{n-1}| \leq |s^1| + 3 \max_{i=n-1,n} |h_i^{-1} (s^0_i - s^0_{i-1})| + |\sigma_2|.$$

Hence, for each  $i = 0, 1, \dots, n$ ,

$$2 |s^1_i| \leq |s^1| + 3 \max_{i=1, \dots, n} |h_i^{-1} (s^0_i - s^0_{i-1})| + |\sigma|.$$

Consequently,

$$2 |s^1| \leq |s^1| + 3 \max_{i=1, \dots, n} |h_i^{-1} (s^0_i - s^0_{i-1})| + |\sigma|,$$

so that

$$|s^1| \leq 3 \max_{i=1, \dots, n} |h_i^{-1} (s^0_i - s^0_{i-1})| + |\sigma|. \tag{4}$$

However, since  $s^0_i = x(t_i)$ , it follows from the mean-value theorem that  $h_i^{-1} (s^0_i - s^0_{i-1}) = x'(\xi_i)$  for some  $\xi_i \in (t_{i-1}, t_i)$ . Hence,  $|h_i^{-1} (s^0_i - s^0_{i-1})| \leq \|x'\|$  for each  $i$ . Putting this into (4) proves (1).

(b) If the mesh  $\pi$  is uniform, then  $h_i = h$  and  $\lambda_i = \mu_i = \frac{1}{2}$  for all  $i$ , and (3) becomes

$$\frac{1}{2} s^1_{i-1} + 2s^1_i + \frac{1}{2} s^1_{i+1} = \frac{3}{2} h^{-1} (s^0_{i+1} - s^0_{i-1}).$$

Using the same argument as before and the fact that  $|s_{i+1}^0 - s_{i-1}^0| \leq 2|s_i^0|$ , inequality (2) follows.

LEMMA 2. *Let  $x \in C$  and  $1 \leq i \leq n$ ; then for each  $t \in [t_{i-1}, t_i]$  we have*

$$|s'(t)| \leq \max\{3h_i^{-1}|s_i^0 - s_{i-1}^0|, |s_{i-1}^1|, |s_i^1|\} \quad (5)$$

and

$$|s'(t)| \leq \frac{3}{2} \max\{3h_i^{-1}|s_{i-1}^0|, 3h_i^{-1}|s_i^0|, |s_{i-1}^1|, |s_i^1|\}. \quad (6)$$

*Proof.* Clearly, on  $I_i = [t_{i-1}, t_i]$  we have

$$s(t) = s_{i-1}^0 M_i(t) + s_i^0 N_i(t) + s_{i-1}^1 P_i(t) + s_i^1 Q_i(t), \quad (7)$$

where

$$\begin{aligned} M_i(t) &= h_i^{-3}(h_i + 2t - 2t_{i-1})(t - t_i)^2, \\ N_i(t) &= h_i^{-3}(h_i - 2t + 2t_i)(t - t_{i-1})^2, \\ P_i(t) &= h_i^{-2}(t - t_{i-1})(t - t_i)^2, \\ Q_i(t) &= h_i^{-2}(t - t_i)(t - t_{i-1})^2. \end{aligned} \quad (8)$$

A simple calculation shows that, on  $I_i$ ,  $M_i' \leq 0$ ,  $N_i' \geq 0$ ,  $M_i' + N_i' = 0$  and  $|N_i'| \leq (3/2)h_i^{-1}$ . Also,  $P_i'$  vanishes at  $t_i$  and  $t_{i-1} + (1/3)h_i$ ,  $Q_i'$  vanishes at  $t_{i-1}$  and  $t_i - (1/3)h_i$ , and  $P_i'(t_{i-1}) = Q_i'(t_i) = 1$ . Thus, by (7),

$$s' = (s_i^0 - s_{i-1}^0)N_i' + s_{i-1}^1 P_i' + s_i^1 Q_i'. \quad (9)$$

Denoting the maxima in (5) and (6) by  $a$  and  $a^*$ , respectively, it follows from (9) that

$$|s'(t)| \leq a \sup\{\phi(t) : t \in I_i\} \quad (10)$$

and

$$|s'(t)| \leq a^* \sup\{\phi^*(t) : t \in I_i\}, \quad (11)$$

where

$$\phi(t) = \frac{1}{3}h_i |N_i'(t)| + |P_i'(t)| + |Q_i'(t)| \quad (12)$$

and

$$\phi^*(t) = \frac{2}{3}h_i |N_i'(t)| + |P_i'(t)| + |Q_i'(t)|. \quad (13)$$

To estimate the above suprema we can assume without loss of generality that  $t_{i-1} = 0$  and  $t_i = 3$ . Then

$$\begin{aligned} \phi(t) &= 1 - \frac{2}{9}t^2 && \text{on } [0, 1], \\ &= -1 + \frac{8}{3}t - \frac{8}{9}t^2 && \text{on } [1, 2], \\ &= -1 + \frac{4}{3}t - \frac{2}{9}t^2 && \text{on } [2, 3], \end{aligned}$$

and  $\phi(0) = \phi(3/2) = \phi(3) = 1$ ,  $\phi(1) = \phi(2) = 7/9$ ,  $\phi'(0) = \phi'(3/2) = \phi'(3) = 0$ . From this it follows that  $\phi \leq 1$  on  $I_i$ . Hence, (10) confirms (5).

By a similar argument we conclude easily that  $\phi^* \leq 3/2$  on  $I_i$ , which, by virtue of (11), proves (6).

In the following, let  $D$  denote the differentiation operator.

LEMMA 3. (a) Let  $x \in C^1_\pi$  and  $\sigma \in R^2$ ; then

$$\|D\mathcal{S}(x, \sigma)\| \leq 3 \|x'\| + |\sigma|. \tag{14}$$

(b) Let  $x \in C$ ,  $\sigma \in R^2$ , and let  $\pi$  be uniform; then

$$\|D\mathcal{S}(x, \sigma)\| \leq \frac{9}{2}h^{-1} \|x\| + |\sigma|. \tag{15}$$

*Proof.* (a) As above,  $h^{-1} |s_i^0 - s_{i-1}^0| \leq \|x'\|$ . Thus, by Lemma 2, (5) and Lemma 1, (1) we have for any  $t \in [a, b]$ ,

$$|s'(t)| \leq \max\{3 \|x'\|, 3 \|x'\| + |\sigma|\} = 3 \|x'\| + |\sigma|$$

which proves (14).

(b) Since  $\mathcal{S}$  is a linear operator on  $C \times R^2$ , it follows that  $\mathcal{S}(x, \sigma) = s + s_*$  with  $s = \mathcal{S}(x, 0)$  and  $s_* = \mathcal{S}(0, \sigma)$ . Now, by (2),  $|s^1| \leq 3h^{-1} \|x\|$  because  $|s^0| \leq \|x\|$ , and by (6),

$$|s'(t)| \leq \frac{3}{2} \max\{3h^{-1} \|x\|, 3h^{-1} \|x\|\} = \frac{9}{2}h^{-1} \|x\|$$

for all  $t \in [a, b]$ . Hence,  $\|s'\| \leq (9/2) h^{-1} \|x\|$ .

Similarly, for  $s_*$  we have by (2),  $|s_*^1| \leq |\sigma|$ . Thus, by (5),  $|s_*'(t)| \leq \max\{0, |\sigma|\} = |\sigma|$  for every  $t \in [a, b]$ . Consequently,  $\|s_*'\| \leq |\sigma|$ .

Finally, by the triangle inequality,  $\|D\mathcal{S}(x, \sigma)\| \leq \|s'\| + \|s_*'\| \leq (9/2) h^{-1} \|x\| + |\sigma|$ , which proves (15).

*Remark 1.* Linearity of  $\mathcal{S}$  and Lemma 3 show readily that, for any  $x \in C'$  and  $\sigma \in R^2$ ,

$$\|D(\mathcal{S}(x, \sigma) - Sx)\| \leq |\sigma - \langle x' \rangle|. \tag{16}$$

Since  $\mathcal{S}(x, \sigma) - Sx$  vanishes at the nodes  $t_i$  of  $\pi$ , it follows by integration that

$$\|\mathcal{S}(x, \sigma) - Sx\| \leq |\sigma - \langle x' \rangle| h. \quad (17)$$

Note that similar results of this kind may be found in [1].

LEMMA 4. *Let  $x \in C_\pi^2$  and  $\sigma \in R^2$ ; then*

$$\|D^2 \mathcal{S}(x, \sigma)\| \leq 3 \|x''\| + 6l^{-1} |\sigma - \langle x' \rangle|. \quad (18)$$

*Proof.* We have [4, p. 11]

$$\mu_i s_{i-1}^2 + 2s_i^2 + \lambda_i s_{i+1}^2 = r_i \quad (19)$$

for  $i = 1, 2, \dots, n-1$ , and

$$\begin{aligned} 2s_0^2 + s_1^2 &= r_0, \\ s_{n-1}^2 + 2s_n^2 &= r_n, \end{aligned} \quad (20)$$

where

$$r_i = 6(h_i + h_{i+1})^{-1} \{h_{i+1}^{-1}(s_{i+1}^0 - s_i^0) - h_i^{-1}(s_i^0 - s_{i-1}^0)\} \quad (21)$$

for  $i = 1, 2, \dots, n-1$ , and

$$\begin{aligned} r_0 &= 6h_1^{-1} \{h_1^{-1}(s_1^0 - s_0^0) - \sigma_1\}, \\ r_n &= 6h_n^{-1} \{\sigma_n - h_n^{-1}(s_n^0 - s_{n-1}^0)\}. \end{aligned} \quad (22)$$

Denoting  $x_i = x(t_i) = s_i^0$  and  $x'_i = x'(t_i)$ ,  $i = 0, 1, \dots, n$ , it follows by Taylor's theorem that

$$\begin{aligned} x_{i+1} &= x_i + h_{i+1} x'_i + \frac{1}{2} h_{i+1}^2 x''(\xi_{i+1}), & \xi_{i+1} &\in (t_i, t_{i+1}), \\ x_{i-1} &= x_i - h_i x'_i + \frac{1}{2} h_i^2 x''(\xi_i), & \xi_i &\in (t_{i-1}, t_i) \end{aligned}$$

for  $i = 1, 2, \dots, n-1$ . Hence, by (21),

$$r_i = 3(h_i + h_{i+1})^{-1} \{h_{i+1} x''(\xi_{i+1}) + h_i x''(\xi_i)\};$$

consequently,

$$|r_i| \leq 3 \|x''\|, \quad i = 1, 2, \dots, n-1. \quad (23)$$

Moreover,  $x_1 = x_0 + h_1 x'_0 + (1/2) h_1^2 x''(\xi_0)$  for some  $\xi_0 \in (t_0, t_1)$ , and consequently,

$$r_0 = 6h_1^{-1}(x'_0 - \sigma_1) + 3x''(\xi_0).$$

Thus

$$|r_0| \leq 3 \|x''\| + 6l^{-1} |\sigma - \langle x' \rangle|; \tag{24}$$

similarly,

$$|r_n| \leq 3 \|x''\| + 6l^{-1} |\sigma - \langle x' \rangle|. \tag{25}$$

On the other hand, (19) and (20) yield

$$\begin{aligned} 2 |s_i^2| &\leq |s^2| + |r_i|, & i = 1, 2, \dots, n-1, \\ 2 |s_0^2| &\leq |s^2| + |r_0|, \\ 2 |s_n^2| &\leq |s^2| + |r_n|. \end{aligned} \tag{26}$$

Hence, using (23)–(25), we get from (26),

$$|s^2| \leq 3 \|x''\| + 6l^{-1} |\sigma - \langle x' \rangle|. \tag{27}$$

Finally, since  $s''$  is a polygon having vertices at the nodes  $t_i$ , we have  $\|s''\| = |s^2|$ . This concludes the proof.

LEMMA 5. *Let  $x \in C^2_\pi$  and  $\sigma \in R^2$ ; then*

$$\|D^3 \mathcal{S}(x, \sigma)\| \leq 6k^{-1} \|x''\| + 12k^{-1} l^{-1} |\sigma - \langle x' \rangle|. \tag{28}$$

*Proof.* Noting the fact that  $s'''$  is constant on each interval  $(t_i, t_{i+1})$  (being a derivative of the polygon  $s''$ ), (28) follows immediately from (27).

For the proof of the theorem we will need the following result due to Hall and Meyer [3].

PROPOSITION 1. *Let  $x \in C^4$ ; then, for  $j = 0, 1, 2, 3$ ,*

$$\|D^j(I - S)x\| \leq H_j \|x^{(4)}\| h^{4-j},$$

where

$$H_0 = \frac{5}{384}, \quad H_1 = \frac{1}{24}, \quad H_2 = \frac{3}{8}, \quad H_3 = \frac{1}{2}(\beta + \beta^{-1}).$$

THEOREM 1. *Let  $p$  be an integer with  $1 \leq p \leq 3$ , and let  $x \in C^{3+p}$ ,  $\omega_1, \dots, \omega_p \in R^2$ . Denote*

$$\begin{aligned} s_1 &= D\mathcal{S}(x, \omega_1), \\ s_2 &= D\mathcal{S}[D\mathcal{S}(x, \omega_1), \omega_2], \\ s_3 &= D\mathcal{S}\{D\mathcal{S}[D\mathcal{S}(x, \omega_1), \omega_2], \omega_3\}, \end{aligned}$$

and let  $d_j = \langle x^{(j)} \rangle - \omega_j$  for  $j = 1, \dots, p$ . Then

(a) for  $p = 1$ ,

$$\|x' - s_1\| \leq \frac{1}{24} \|x^{(4)}\| h^3 + |d_1|, \quad (29)$$

(b) for  $p = 2$ ,

$$\|x'' - s_2\| \leq \frac{1}{24} \|x^{(5)}\| h^3 + \frac{9}{8} \|x^{(4)}\| h^2 + |d_2| + 18l^{-1} |d_1|, \quad (30)$$

(c) for  $p = 3$ ,

$$\begin{aligned} \|x''' - s_3\| &\leq \frac{1}{24} \|x^{(6)}\| h^3 + \frac{9}{8} \|x^{(5)}\| h^2 + \frac{63}{4} \|x^{(4)}\| h^2 k^{-1} \\ &\quad + |d_3| + 18l^{-1} |d_2| + 108(k^{-1} + l^{-1}) l^{-1} |d_1|. \end{aligned} \quad (31)$$

If, in addition, the mesh  $\pi$  is uniform, then

$$\|x^{(p)} - s_p\| \leq \frac{1}{24} \sum_{j=1}^p \left(\frac{9}{2}\right)^{p-j} \|x^{(3+j)}\| h^{3+j-p} + \sum_{j=1}^p \left(\frac{9}{2}\right)^{p-j} |d_j| h^{j-p}. \quad (32)$$

*Proof.* (a) Using Proposition 1 and Lemma 3, we have by the triangle inequality an linearity of  $\mathcal{S}$ ,

$$\begin{aligned} \|x' - s_1\| &\leq \|D(x - Sx)\| + \|D[\mathcal{S}(x, \langle x' \rangle) - \mathcal{S}(x, \omega_1)]\| \\ &= \|D(I - S)x\| + \|D\mathcal{S}(0, \langle x' \rangle - \omega_1)\| \leq \frac{1}{24} \|x^{(4)}\| h^3 + |\langle x' \rangle - \omega_1| \end{aligned}$$

which proves (29).

(b) By linearity of  $\mathcal{S}$  we have the identity  $x'' - s_2 = A_1 + A_2$ , where

$$A_1 = D[Dx - \mathcal{S}(Dx, \omega_2)], \quad A_2 = D\mathcal{S}\{D(x - \mathcal{S}(x, \omega_1)), 0\}.$$

Due to Proposition 1 and Lemma 3,

$$\begin{aligned} \|A_1\| &\leq \|D[Dx - S(Dx)]\| + \|D[\mathcal{S}(Dx, \langle x'' \rangle) - \mathcal{S}(Dx, \omega_2)]\| \\ &= \|D(I - S)(Dx)\| + \|D\mathcal{S}(0, \langle x'' \rangle - \omega_2)\| \leq \frac{1}{24} \|x^{(5)}\| h^3 + |\langle x'' \rangle - \omega_2|. \end{aligned}$$



Moreover, Lemmas 3 and 4 imply that

$$\begin{aligned} \|A_2\| &\leq 3 \|D^2(x - \mathcal{S}(x, \omega_1))\| \leq 3 \|D^2(x - Sx)\| \\ &\quad + 3 \|D^2\mathcal{S}(x, \langle x' \rangle) - \mathcal{S}(x, \omega_1)\| \\ &= 3 \|D^2(I - S)x\| + 3 \|D^2\mathcal{S}(0, \langle x' \rangle - \omega_1)\| \\ &\leq \frac{9}{8} \|x^{(4)}\| h^2 + 18l^{-1} |\langle x' \rangle - \omega_1|. \end{aligned}$$

Introducing this into  $\|x'' - s_2\| \leq \|A_1\| + \|A_2\|$ , (30) follows.

(c) We have the identity  $x''' - s_3 = B_1 + B_2 + B_3$ , where

$$\begin{aligned} B_1 &= D[D^2x - \mathcal{S}(D^2x, \omega_3)], \\ B_2 &= D\mathcal{S}\{D[Dx - \mathcal{S}(Dx, \omega_2)], 0\}, \\ B_3 &= D\mathcal{S}\{D\mathcal{S}[D(x - \mathcal{S}(x, \omega_1)), 0], 0\}. \end{aligned}$$

Using Proposition 1 and Lemmas 3 and 4, we get

$$\begin{aligned} \|B_1\| &\leq \|(I - S)D^2x\| + \|D\mathcal{S}(0, \langle x''' \rangle - \omega_3)\| \\ &\leq \frac{1}{24} \|x^{(6)}\| h^3 + |\langle x''' \rangle - \omega_3|. \end{aligned} \tag{33}$$

Similarly,

$$\begin{aligned} \|B_2\| &\leq 3 \|D^2[Dx - \mathcal{S}(Dx, \omega_2)]\| \\ &\leq 3 \|D^2(I - S)Dx\| + 3 \|D^2\mathcal{S}(0, x'' - \omega_2)\| \\ &\leq \frac{9}{8} \|x^{(5)}\| h^2 + 18l^{-1} |\langle x'' \rangle - \omega_2|. \end{aligned}$$

To estimate  $\|B_3\|$ , put  $u = D(x - \mathcal{S}(x, \omega_1))$ . Then

$$\|B_3\| \leq 3 \|D^2\mathcal{S}[u, 0]\| \leq 3\{3 \|u''\| + 6l^{-1} |\langle u' \rangle|\}. \tag{34}$$

However,

$$\begin{aligned} \|u''\| &= \|D^3(x - \mathcal{S}(x, \omega_1))\| \leq \|D^3(I - S)x\| + \|D^3\mathcal{S}(0, \langle x' \rangle - \omega_1)\| \\ &\leq \frac{1}{2}(\beta + \beta^{-1}) \|x^{(4)}\| h + 12k^{-1}l^{-1} |\langle x' \rangle - \omega_1|. \end{aligned} \tag{35}$$

On the other hand,

$$\begin{aligned} |\langle u' \rangle| &\leq \|u'\| = \|D^2(x - \mathcal{S}(x, \omega_1))\| \leq \|D^2(I - S)x\| \\ &\quad + \|D^2\mathcal{S}(0, \langle x' \rangle - \omega_1)\| \leq \frac{3}{8} \|x^{(4)}\| h^2 + 6l^{-1} |\langle x' \rangle - \omega_1|. \end{aligned} \tag{36}$$

Introducing (35) and (36) into (34), we obtain

$$\|B_3\| \leq \left\{ \frac{9}{2} (\beta + \beta^{-1}) h + \frac{27}{4} h^2 l^{-1} \right\} \|x^{(4)}\| + 108l^{-1}(k^{-1} + l^{-1}) |d_1|. \quad (37)$$

Finally, it is easy to see that the coefficient  $\{\dots\}$  of  $\|x^{(4)}\|$  in (37) does not exceed  $(63/4) h^2 k^{-1}$ . Thus, making use of the inequality  $\|x''' - s_3\| \leq \|B_1\| + \|B_2\| + \|B_3\|$ , we arrive readily at (31).

To prove (32), we use Proposition 1, the above identities for  $x^{(p)} - s_p$  ( $p = 1, 2, 3$ ), and repeatedly apply inequality (15) in Lemma 3. Since the procedure is straightforward, we omit the details.

*Remark 2.* An alternative spline-on-spline approximant to  $x^{(3)}$  is furnished by  $\tilde{s}_3 = D^2 \mathcal{S}[D\mathcal{S}(x, \omega_1), \omega_2]$ . The argument used in the proof of Theorem 1 will convince us that for  $x \in C^5$  and  $\omega_1, \omega_2 \in R^2$ , we have

$$\begin{aligned} \|x^{(3)} - \tilde{s}_3\| &\leq \frac{3}{8} \|x^{(5)}\| h^2 + \frac{21}{4} \|x^{(4)}\| h^2 k^{-1} + 6l^{-1} |\langle x'' \rangle - \omega_2| \\ &\quad + 36(k^{-1} + l^{-1}) l^{-1} |\langle x' \rangle - \omega_1|. \end{aligned}$$

### 3. DISCRETE TYPE ESTIMATES

In this section we will assume that the mesh  $\pi$  is uniform. For  $x \in C$  we put  $\hat{x} = (x(t_0), x(t_1), \dots, x(t_n)) \in R^{n+1}$ .

Define the operator  $\mathcal{D}: R^{n+1} \times R^2 \rightarrow R^{n+1}$  as follows: for  $\xi = (\xi_0, \xi_1, \dots, \xi_n) \in R^{n+1}$  and  $\sigma = (\sigma_1, \sigma_2) \in R^2$ , let  $\mathcal{D}(\xi, \sigma) = (\widehat{s'})$ , where  $s$  is the unique cubic spline associated with  $\pi$  such that  $s(t_i) = \xi_i$  for  $i = 0, 1, \dots, n$  and  $s'(t_0) = \sigma_1, s'(t_n) = \sigma_2$ .

Observe that, for  $x \in C, \mathcal{D}(\hat{x}, \sigma) = \widehat{(\mathcal{S}'(x, \sigma))}$ .

Clearly,  $\mathcal{D}$  is a linear operator on  $R^{n+1} \times R^2$ , and by Lemma 1 we have

$$|\mathcal{D}(\xi, \sigma)| \leq 3 |\xi| h^{-1} + |\sigma|. \quad (38)$$

We will use the following assertion (for the proof see [2, p. 212], or [5, p. 69]).

**PROPOSITION 2.** *Let  $x \in C^5$  and let the mesh  $\pi$  be uniform. Then*

$$|\mathcal{D}(\hat{x}, \langle x' \rangle) - \widehat{\langle x' \rangle}| \leq \frac{1}{60} \|x^{(5)}\| h^4. \quad (39)$$

Note that (38), (39), linearity of  $\mathcal{D}$  and the triangle inequality imply that for  $x \in C^5$  and  $\sigma \in R^2$ ,

$$|\mathcal{D}(\hat{x}, \sigma) - \widehat{(x')}| \leq \frac{1}{60} \|x^{(5)}\| h^4 + |\sigma - \langle x' \rangle|. \tag{40}$$

**THEOREM 2.** *Let  $p$  be an integer with  $1 \leq p \leq 4$ , and let  $x \in C^{4+p}$ ,  $\omega_1, \dots, \omega_p \in R^2$ . Moreover, let the mesh  $\pi$  be uniform. Denote*

$$\begin{aligned} \eta_1 &= \mathcal{D}(\hat{x}, \omega_1), \\ \eta_2 &= \mathcal{D}[\mathcal{D}(\hat{x}, \omega_1), \omega_2], \\ \eta_3 &= \mathcal{D}\{\mathcal{D}[\mathcal{D}(x, \omega_1), \omega_2], \omega_3\}, \\ \eta_4 &= \mathcal{D}[\mathcal{D}\{\mathcal{D}[\mathcal{D}(\hat{x}, \omega_1), \omega_2], \omega_3\}, \omega_4], \end{aligned}$$

and let  $d_j = \langle x^{(j)} \rangle - \omega_j$  for  $j = 1, \dots, p$ . Then

$$|\widehat{(x^{(p)})} - \eta_p| \leq \frac{1}{60} \sum_{j=1}^p 3^{p-j} \|x^{(4+j)}\| h^{4+j-p} + \sum_{j=1}^p 3^{p-j} |d_j| h^{j-p}. \tag{41}$$

*Proof.* For  $p = 1$ , inequality (41) reduces to (40). Furthermore, by virtue of the linearity of  $\mathcal{D}$  the following equalities are true:

$$\eta_2 - \widehat{(x'')} = \mathcal{D}[\mathcal{D}(\hat{x}, \omega_1) - \widehat{(x')}, 0] + \mathcal{D}[\widehat{(x')}, \omega_2] - \widehat{(x'')}, \tag{42}$$

$$\begin{aligned} \eta_3 - \widehat{(x^{(3)})} &= \mathcal{D}\{\mathcal{D}[\mathcal{D}(\hat{x}, \omega_1) - \widehat{(x')}, 0], 0\} + \mathcal{D}\{\mathcal{D}[\widehat{(x')}, \omega_2] - \widehat{(x'')}, 0\} \\ &\quad + \mathcal{D}\{\widehat{(x'')}, \omega_3\} - \widehat{(x^{(3)})}, \end{aligned} \tag{43}$$

$$\begin{aligned} \eta_4 - \widehat{(x^{(4)})} &= \mathcal{D}[\mathcal{D}\{\mathcal{D}[\mathcal{D}(\hat{x}, \omega_1) - \widehat{(x')}, 0], 0\}, 0] \\ &\quad + \mathcal{D}[\mathcal{D}\{\mathcal{D}[\widehat{(x')}, \omega_2] - \widehat{(x'')}, 0\}, 0] \\ &\quad + \mathcal{D}[\mathcal{D}\{\widehat{(x'')}, \omega_3\} - \widehat{(x^{(3)})}, 0] + \mathcal{D}[\widehat{(x^{(3)})}, \omega_4] - \widehat{(x^{(4)})}. \end{aligned} \tag{44}$$

Applying the triangle inequality to (42) through (44) and using (38) with (40), we obtain (41) for  $p = 2, 3, 4$ . Hence the proof.

**Remark 3.** An alternative approximant to  $\widehat{(x^{(4)})}$  can be constructed as follows: for  $x \in C^6$ , let  $y = \mathcal{D}\mathcal{S}[D\mathcal{S}(x, \langle x' \rangle), \langle x'' \rangle]$  and put  $z = \mathcal{S}(y, \langle x^{(3)} \rangle)$ . Then it can be easily shown that

$$|\widehat{(x^{(4)})} - \widehat{(z'')}| \leq \frac{23}{40} \|x^{(6)}\| h^2 + \frac{3}{5} \|x^{(5)}\| h.$$

## 4. RESULTS OF SOME NUMERICAL EXPERIMENTS

Let us define the maximal normalized errors at the knots in the second and fourth derivative of a given function  $x$  computed by the spline-on-spline technique as follows:

$$\varepsilon^{(2)} = \|x''\|^{-1} \cdot |\eta_2 - \widehat{(x'')}|$$

with

$$\omega_1 = \langle x' \rangle, \quad \omega_2 = \langle x'' \rangle,$$

and

$$\varepsilon^{(4)} = \|x^{(4)}\|^{-1} \cdot |(z'') - \widehat{(x^{(4)})}|,$$

where  $\eta_2$  and  $z$  are defined in Theorem 2 and Remark 3, respectively.

Also, to test the suitability of our technique to points other than the knots, we define the maximal normalized error in the second derivative at the set of points  $\tau_i = t_i + (1/3)h$ ,  $i = 0, 1, \dots, n-1$  by

$$\hat{\varepsilon}^{(2)} = \|x''\|^{-1} \cdot \max_{0 \leq i \leq n-1} \{|D^2x - D\mathcal{L}(DSx, \langle x'' \rangle)\}(\tau_i)|.$$

The results of some computational experiments are given in Table I for the functions  $e^t$ ,  $e^{5t}$  and  $\sin 5t$  with  $0 \leq t \leq 1$ . We have chosen for  $h$  the values  $10^{-1}$ ,  $10^{-2}$  and  $10^{-3}$ . It is evident from the table that  $\varepsilon^{(2)}$ ,  $\varepsilon^{(4)}$  and  $\hat{\varepsilon}^{(2)}$  are, respectively,  $O(h^3)$ ,  $O(h)$  and  $O(h^2)$ , which agrees with our theoretical error bounds.

TABLE I  
ERRORS IN THE SECOND AND FOURTH DERIVATIVES

	$h$	$e^t$	$e^{5t}$	$\sin 5t$
$\varepsilon^{(2)}$	$10^{-1}$	$3.219 \times 10^{-6}$	$4.900 \times 10^{-4}$	$9.100 \times 10^{-4}$
	$10^{-2}$	$4.326 \times 10^{-9}$	$4.735 \times 10^{-7}$	$5.548 \times 10^{-7}$
	$10^{-3}$	$4.613 \times 10^{-12}$	$5.496 \times 10^{-10}$	$5.580 \times 10^{-10}$
$\varepsilon^{(4)}$	$10^{-1}$	$2.987 \times 10^{-3}$	$2.239 \times 10^{-2}$	$2.790 \times 10^{-2}$
	$10^{-2}$	$2.395 \times 10^{-4}$	$1.337 \times 10^{-3}$	$1.224 \times 10^{-3}$
	$10^{-3}$	$2.333 \times 10^{-5}$	$1.220 \times 10^{-4}$	$1.243 \times 10^{-4}$
$\hat{\varepsilon}^{(2)}$	$10^{-1}$	$9.535 \times 10^{-6}$	$8.759 \times 10^{-4}$	$1.757 \times 10^{-3}$
	$10^{-2}$	$1.011 \times 10^{-8}$	$1.232 \times 10^{-6}$	$1.760 \times 10^{-6}$
	$10^{-3}$	$9.296 \times 10^{-12}$	$1.269 \times 10^{-9}$	$1.760 \times 10^{-9}$

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